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ON ABSTRACT VOLTERRA EQUATIONS WITH KERNELS HAVING A POSITIVE RESOLVENT

BY

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ABSTRACT

We consider the nonlinear abstract Volterra equation of convolution type:

$$\text{(V)} \qquad u(t)+b*Au(t)=u_0+b*g(t), \quad t \geqq 0,$$

where A is m-accretive in a Banach space X, b is a given real kernel, u_0 and g are given. Boundedness and asymptotic properties of the solutions are established under the assumption that the kernel satisfies certain natural positivity conditions.

1. Introduction

Let X be a real Banach space with norm $\|\cdot\|$. Let A be a m-accretive operator in X, [3], i.e. for every $\lambda > 0$, $J_\lambda := (I+\lambda A)^{-1}$ is a nonexpansive map which is everywhere defined on X. We consider the following Volterra equation of convolution type:

$$(1.1) \qquad u(t)+b*Au(t) \ni f(t), \qquad t \geqq 0$$

where b is a given real kernel, f is a given function with values in X and $b*Au(t)=\int_0^t b(t-s)Au(s)ds$. Since for every $\lambda > 0$, the Yosida approximation of A, $A_\lambda := \lambda^{-1}(I-J_\lambda)$ is Lipschitz continuous, the equation

$$(1.1)_\lambda \qquad u(t)+b*A_\lambda u(t)=f(t), \qquad t \geqq 0$$

possesses a unique solution $u_\lambda \in C([0,T];X)$ if $b \in L^1[0,T]$ and $f \in C([0,T];X)$, $T>0$. In [4], Crandall and Nohel have proved that if the assumption

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$$
\text{(H1)} \qquad \begin{cases} b \in W^{1,1}[0,T], \quad b(0) > 0, \quad \dot{b} \in BV[0,T] \\ \quad f \in W^{1,1}[0,T;X], \quad f(0) \in \overline{D(A)} \end{cases}
$$

is satisfied, then there exists $u \in C([0,T];X)$ such that $\lim_{\lambda \downarrow 0} u_\lambda = u$ in $C([0,T];X)$; u is called the *generalized solution* of (1.1). Note that if (H1) is satisfied, then there exists a unique $u_0 \in \overline{D(A)}$ and a unique $g \in L^1(0,T:X)$ such that

$$
(1.2) \qquad f(t) = u_0 + b * g(t), \qquad 0 \leqq t \leqq T.
$$

Indeed $u_0 = f(0)$ and g is the unique solution of the equation

$$
b(0)g(t) + \dot{b} * g(t) = \dot{f}(t), \qquad 0 \leqq t \leqq T
$$

(where $\cdot = d/dt$). Conversely, if $b \in W^{1,1}[0,T]$, $b(0) > 0$, $\dot{b} \in BV[0,T]$ and $u_0 \in \overline{D(A)}$, $g \in L^1(0,T;X)$, then f given by (1.2) satisfies assumption (H1).

The proof in [4] of the existence of a generalized solution of (1.1) shows that (1.1) is closely related to the equation

$$
(1.3) \qquad \begin{cases} \dot{u}(t) + Au(t) \ni g(t), \qquad 0 < t \leqq T, \\ \qquad u(0) = u_0, \end{cases}
$$

which is (1.1) with $b \equiv 1$. It is known [1], that if u_1 and u_2 are generalized solutions of (1.3) corresponding to the data $u_{0,1}$, $u_{0,2}$ and g_1, g_2, then the following estimate, which implies continuous dependence of solutions of (1.3), holds:

$$
(1.4) \qquad \|u_1(t) - u_2(t)\| \leqq \|u_{0,1} - u_{0,2}\| + b * \|g_1 - g_2\|(t)
$$

on $[0,T]$, with $b \equiv 1$. In this paper we consider a class of kernels satisfying (H1), containing the kernel $b \equiv 1$, for which the estimate (1.4) still holds. Such class of kernels was introduced in [2, assumptions H4, H5]. Moreover, we prove that if the kernel b belongs to this class and is in $L^1(0,\infty)$, then the generalized solution of (1.1) converges strongly to a limit u_∞ provided that g itself is bounded and converges to a limit g_∞. If $b \notin L^1(0,\infty)$, it is well-known that u may not converge to a limit. (Take $X = \mathbf{R}^2$ with the Euclidean norm,

$$
A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

$b \equiv 1$, $g = 0$, $u_0 \neq 0$). Work is in progress on an analogous result in the case $b \notin L^1(0,\infty)$ and $A = \omega I + B$ ($\omega > 0$, B m-accretive).

In order to state our main assumption on the kernel b we need the following definitions. For $b \in L^1(0,T)$, let us denote by $r(b)$ the resolvent of b, i.e. the unique solution in $L^1(0,T)$ of the equation

$$(1.5) \qquad r + b * r = b, \qquad 0 \leqq t \leqq T,$$

and by $s(b)$, the unique solution in $AC[0,T]$ of the equation

$$(1.6) \qquad s + b * s = 1, \qquad 0 \leqq t \leqq T.$$

Our basic assumption on the kernel b is

$$\text{(H2)} \qquad \begin{cases} \text{For every } \lambda > 0,\ r(\lambda b) \geqq 0 \text{ a.e. on } [0,T] \\ \qquad \text{and } s(\lambda b) \geqq 0 \text{ on } [0,T]. \end{cases}$$

It is known [7], [5], [2] that if $b \in L^1(0,T)$, is positive, nonincreasing and if $\log b$ is convex on $(0,T)$, then b satisfies (H2). Observe that if b is completely monotonic on $(0,\infty)$, then $\log b$ is convex [7]. Observe also that (H2) implies $b \geqq 0$. In order to avoid trivialities we shall assume that b is not identically equal to 0. In connection with this class of kernels we mention the following "positivity" result:

THEOREM [2; theorem 5]. *Let b, f satisfy* (H1) *and* (H2) *on* $[0,T]$ *with* $f = u_0 + b * g$. *Let P be a closed convex cone in X. If $J_\lambda(P) \subseteq P$ for every $\lambda > 0$, $u_0 \in P$ and $g(t) \in P$* a.e. *on* $[0,T]$, *then u the generalized solution of* (1.1) *satisfies* $u(t) \in P$, $t \in [0,T]$.

2. Statement of results

We first give the generalization of (1.4) to (1.1) with kernels b satisfying (H2).

THEOREM 1. *Let b, f_1, f_2 satisfy* (H1) *and* (H2) *on* $[0,T]$, *with* $f_i = u_{0,i} + b * g_i$, $i = 1,2$. *Let u_1, u_2 be the corresponding generalized solutions of* (1.1) *on* $[0,T]$. *Then*

$$(2.1) \qquad \|u_1(t) - u_2(t)\| \leqq \|u_{0,1} - u_{0,2}\| + b * \|g_1 - g_2\|(t)$$

$0 \leqq t \leqq T$ *holds.*

Our main result concerns the asymptotic behaviour of solutions of (1.1) as $t \to \infty$. For results in this direction, in the scalar case, but for more general kernels b, we refer the reader to [6].

THEOREM 2. *Let b, f satisfy* (H1) *and* (H2) *on* $[0, T]$ *for every* $T > 0$, *with* $f = u_0 + b * g$ *and* $b \not\equiv 0$. *If* $b \in L^1(0, \infty)$, $g \in L^\infty(\mathbf{R}^+, X)$ *and* $\lim_{t \to \infty} g(t) = g_\infty$ *exists in X, then*

$$\|u(t) - u_\infty\| \leq \frac{\int_t^\infty b(s)ds}{\int_0^\infty b(s)ds} \|u_0 - u_\infty\| + b * \|g - g_\infty\|(t) \tag{2.2}$$

holds for $t > 0$, *where u is the generalized solution of* (1.1) *and* $u_\infty = (I + \bar{b}A)^{-1}(u_0 + \bar{b}g_\infty)$ *with* $\bar{b} = \int_0^\infty b(s)ds$.

3. Proofs

In the proofs we shall use the fact that if $v \in L^1(0, T; X)$ satisfies

$$v(t) + b * v(t) = u_0 + b * g(t), \qquad 0 \leq t \leq T \tag{3.1}$$

with $b \in L^1(0, T)$, $u_0 \in X$ and $g \in L^1(0, T; X)$ then

$$v(t) = s(b)(t)u_0 + r(b) * g(t), \qquad 0 \leq t \leq T \tag{3.2}$$

holds.

PROOF OF THEOREM 1. We first establish (2.1) with A replaced by A_λ, $\lambda > 0$ and then we pass to the limit as $\lambda \downarrow 0$. For $\lambda > 0$, let $u_{i,\lambda}$ satisfy

$$u_{i,\lambda} + b * A_\lambda u_{i,\lambda} = u_{0,i} + b * g_i, \qquad t \in [0, T], \quad i = 1, 2. \tag{3.3}$$

From the definition of A_λ, we have

$$u_{i,\lambda} + \lambda^{-1} b * u_{i,\lambda} = \lambda^{-1} b * J_\lambda u_{i,\lambda} + u_{0,\lambda} + b * g_i, \qquad i = 1, 2. \tag{3.4}$$

Using (3.2) we get

$$u_{i,\lambda} = r(\lambda^{-1}b) * J_\lambda u_{i,\lambda} + s(\lambda^{-1}b)u_{0,i} + \lambda r(\lambda^{-1}b) * g_i, \qquad i = 1, 2. \tag{3.5}$$

Next we estimate $\|u_{1,\lambda} - u_{2,\lambda}\|$. Since J is nonexpansive, $s(\lambda^{-1}b)$ and $r(\lambda^{-1}b)$ are nonnegative, we obtain:

$$\begin{aligned} \|u_{1,\lambda} - u_{2,\lambda}\| &\leq r(\lambda^{-1}b) * \|u_{1,\lambda} - u_{2,\lambda}\| + s(\lambda^{-1}b)\|u_{0,1} - u_{0,2}\| \\ &\quad + \lambda r(\lambda^{-1}b) * \|g_1 - g_2\|. \end{aligned} \tag{3.6}$$

We take the convolution of (3.6) with $\lambda^{-1}b$ (which is nonnegative) and we add (3.6). We have

$$
\begin{aligned}
&\|u_{1,\lambda}-u_{2,\lambda}\|+\lambda^{-1}b*\|u_{1,\lambda}-u_{2,\lambda}\| \\
&\quad \leqq (r(\lambda^{-1}b)+\lambda^{-1}b*r(\lambda^{-1}b))*\|u_{1,\lambda}-u_{2,\lambda}\| \\
&\qquad +(s(\lambda^{-1}b)+\lambda^{-1}b*s(\lambda^{-1}b))\|u_{0,1}-u_{0,2}\| \\
&\qquad +\lambda(r(\lambda^{-1}b)+\lambda^{-1}b*r(\lambda^{-1}b))*\|g_1-g_2\|.
\end{aligned} \tag{3.7}
$$

From the definition of $r(\lambda^{-1}b)$ and $s(\lambda^{-1}b)$, we obtain $\|u_{1,\lambda}-u_{2,\lambda}\|\leqq \|u_{0,1}-u_{0,2}\|+b*\|g_1-g_2\|$. The conclusion of Theorem 1 follows by letting λ go to 0.

Proof of Theorem 2. As in the proof of Theorem 1, we first prove the result with A replaced by A_λ, $\lambda>0$ and then we pass to the limit as $\lambda\downarrow 0$.

For $\lambda>0$, let u_λ satisfy

$$u_\lambda+b*A_\lambda u_\lambda=u_0+b*g. \tag{3.8}$$

From the definition of A_λ and (3.2) we have:

$$u_\lambda=r(\lambda^{-1}b)*J_\lambda u_\lambda+s(\lambda^{-1}b)u_0+\lambda r(\lambda^{-1}b)*g. \tag{3.9}$$

Since A is m-accretive, A_λ *is also* m-accretive and there is a unique $u_{\lambda\infty}$ satisfying

$$u_{\lambda\infty}+\bar{b}A_\lambda u_{\lambda\infty}=u_0+\bar{b}g_\infty \tag{3.10}$$

where $\bar{b}=\int_0^\infty b(s)ds$.

Using again the fact that $b\in L^1(0,\infty)$, we can rewrite (3.10) as

$$u_{\lambda\infty}+b*A_\lambda u_{\lambda\infty}=u_0+b*g+b*(g_\infty-g)-\xi w_\lambda \tag{3.11}$$

where

$$\xi(t):=\int_t^\infty b(s)ds \tag{3.12}$$

and

$$w_\lambda:=A_\lambda u_{\lambda\infty}-g_\infty. \tag{3.13}$$

Let η satisfy

$$\eta+\lambda^{-1}b*\eta=\xi. \tag{3.14}$$

Then obviously ηw_λ satisfies

$$\eta w_\lambda+\lambda^{-1}b*\eta w_\lambda=\xi w_\lambda. \tag{3.15}$$

Using (3.11), (3.15), (3.2) and the definition of A_λ we obtain

$$u_{\lambda\infty} = r(\lambda^{-1}b) * J_\lambda u_{\lambda\infty} + s(\lambda^{-1}b)u_0 + \lambda r(\lambda^{-1}b) * g + \lambda r(\lambda^{-1}b) * (g_\infty - g) - \eta w_\lambda. \tag{3.16}$$

Subtracting (3.16) from (3.9) and using the fact that J_λ is nonexpansive, $s(\lambda^{-1}a)$, $r(\lambda^{-1}a)$ are nonnegative, we get:

$$\| u_\lambda - u_{\lambda\infty} \| \leqq r(\lambda^{-1}b) * \| u_\lambda - u_{\lambda\infty} \| + \lambda r(\lambda^{-1}b) * \| g_\infty - g \| + |\eta| \, \| w_\lambda \|. \tag{3.17}$$

Next we take the convolution of (3.17) with $\lambda^{-1}b$ (which is nonnegative) and we add (3.17); we obtain

$$\| u_\lambda - u_{\lambda\infty} \| \leqq b * \| g - g_\infty \| + (|\eta| + \lambda^{-1}b * |\eta|) \| w_\lambda \|. \tag{3.18}$$

We claim that η is nonnegative. Indeed η satisfies (3.14) with $\xi(t) = \bar{b} - \int_0^t b(s)ds$. Thus η satisfies (3.1) for every $T > 0$, with $X = \mathbf{R}$, b replaced by $\lambda^{-1}b$, u_0 replaced by $\bar{b}$ and g replaced by $-\lambda \mathbf{1}$, where $\mathbf{1}(t) \equiv 1$. From (3.2) we get

$$\eta(t) = s(\lambda^{-1}b)(t)\bar{b} - \lambda \int_0^t r(\lambda^{-1}b)(\tau)d\tau, \qquad t > 0. \tag{3.19}$$

By using the identity

$$s(\lambda^{-1}b)(t) + \int_0^t r(\lambda^{-1}b)(\tau)d\tau = 1, \qquad t \geqq 0 \tag{3.20}$$

we have

$$\dot{\eta}(t) = -\bar{b}r(\lambda^{-1}b)(t) - \lambda r(\lambda^{-1}b)(t), \qquad t \geqq 0. \tag{3.21}$$

The fact that $\bar{b}, \lambda$ are positive and assumption (H2) imply that η is nonincreasing. It remains to prove that $\lim_{t\to\infty} \eta(t) \geqq 0$.

Form (3.20) and assumption (H2), it follows that $r(\lambda^{-1}b) \in L^1(0,\infty)$. (3.14) implies

$$\eta = \xi - r(\lambda^{-1}b) * \xi. \tag{3.22}$$

Hence $\lim_{t\to\infty} \eta(t) = 0$ and $\eta(t) \geqq 0$ for every $t \geqq 0$. Replacing $|\eta|$ by η in (3.18) and using (3.14), we obtain:

$$\| u_\lambda - u_{\lambda\infty} \| \leqq \xi \| w_\lambda \| + b * \| g - g_\infty \|. \tag{3.23}$$

Since $\bar{b} > 0$, using (3.13) we have

$$\xi(t)\|w_\lambda\| = \frac{\xi(t)}{\bar{b}}\|\bar{b}A_\lambda u_{\lambda\infty} - \bar{b}g_\infty\|. \tag{3.24}$$

Finally using (3.10), (3.12), (3.23) and (3.24) we get

$$\|u_\lambda(t) - u_{\lambda\infty}\| \leqq \frac{\int_t^\infty b(s)ds}{\int_0^\infty b(s)ds}\|u_0 - u_{\lambda\infty}\| + b * \|g - g_\infty\|(t), \qquad t \geqq 0. \tag{3.25}$$

Observe that (3.25) is the conclusion of the theorem with A replaced by A_λ.

Since A is m-accretive we have

$$\lim_{\lambda\downarrow 0} u_{\lambda\infty} := \lim_{\lambda\downarrow 0}(I + \bar{b}A_\lambda)^{-1}(u_0 + \bar{b}g_\infty)$$

$$= \lim_{\lambda\downarrow 0}(I + \bar{b}A)^{-1}(u_0 + \bar{b}g_\infty) =: u_\infty.$$

Using assumption (H1), $\lim_{\lambda\downarrow 0} u_\lambda = u$ in $C([0,T];X)$, thus (2.2) follows from (3.25) by letting λ go to 0.

Remark. It is clear from the proofs of the theorems that the assumption (H1) has been used only to insure that $\lim_{\lambda\downarrow 0} u_\lambda$ exists in $C([0,T];X)$, for every $T>0$. Indeed Theorems 1 and 2 are valid for A replaced by A_λ, $\lambda>0$, if the assumption (H1) is replaced by the assumption

$$\text{(H1')} \qquad \begin{cases} a \in L^1(0,T), \\ u_0 \in X, \qquad g \in L^1(0,T;X). \end{cases}$$

It has been proved in [2, theorem 1(ii), theorem 2(i), remark 2.3] that under the assumptions (H1') and (H2), if A is linear m-accretive with $D(A)$ dense in X that $\lim_{\lambda\downarrow 0} u_\lambda$ exists in $L^1(0,T;X)$. Therefore in the linear case, Theorems 1 and 2 are true with (H1) replaced by (H1'). Then pointwise inequalities (2.1) and (2.2) have to be replaced by a.e. inequalities.

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